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Laplace–Runge–Lenz vector, ladder operators and supersymmetry

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Abstract
We show that the Laplace–Runge–Lenz vector (LRLV) generates the Infeld–Hull radial factorization and the pair of isospectral Hamiltonians for the non-relativistic Kepler–Coulomb quantum problem. To do this we only use the LRLV and arguments of general soundness. Finally, the well known restrictions on the orbital angular momentum and the principal quantum numbers are rederived from the corresponding ladder operators.

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1. Introduction

Since the early days of quantum mechanics, the Schrödinger equation has been solved for a large number of potentials employing a variety of approaches. The solutions of certain potentials have been given using the constants of motion of the problem [1, 2] and using the factorization method [3], for example. Although the methods of solution usually focus on different aspects of solvable potentials, they are not independent from each other. In fact, supersymmetric quantum mechanics (SUSYQM) [4] is related [5] to the factorization method and some algebraic approaches [6]. Also, for some systems, SUSY charges have been investigated in connection with the constants of motion of the problem [7, 8].

It is well known that the non-relativistic Kepler–Coulomb quantum problem (NRKCQP) can be solved with its constants of motion [1] and by factorization techniques [3]. The Laplace–Runge–Lenz vector (LRLV), $\boldsymbol{A}$, is the one responsible for the accidental degeneracy of the hydrogen atom's energy levels with respect to the orbital angular momentum quantum number, ℓ [9]. The radial ladder operators for the reduced wavefunction $f_{n\ell}(r)$, acting on ℓ, obtained by the Infeld–Hull (IH) factorization method have been related to the operator $\boldsymbol{A}$ [7, 10, 11]. The first work in this direction was that of Coish [10]. He used Pauli operators and found that the vectors $\boldsymbol{L}$ and $\boldsymbol{A}$ obey the $so(4)$ algebra [12]. Moreover Biedenharn and Louck in [11] derived some results of [10]. To do this they considered the electron as a Pauli particle. They also employed Dirac radial decomposition as well as the properties of the Dirac operator $\mathcal{K} = -(\boldsymbol{\sigma} \cdot \boldsymbol{L} + 1)$, and those of the operator $\boldsymbol{\sigma} \cdot \boldsymbol{A}$. In [7] Tangerman and Tjon identified the SUSY charges by using the spin degrees of freedom of the electron. It can be concluded from the previous works that the spin operators and their properties seems to be essential to obtain their results.

In this paper we show the relation between $\boldsymbol{A}$ and the supersymmetric approach of the NRKCQP from a different point of view. We show that the IH factorization and the supersymmetry of the problem is only generated by the LRLV and arguments of general soundness. By doing this we apply the spherical components of $\boldsymbol{A}$ to any state wavefunction of the full function space spanned by the solutions of the Schrödinger equation. An idea similar to ours has been simply suggested by Grosse [13].

In the next section, for the NRKCQP we apply the spherical components $A_\pm$ and A_0 of $\boldsymbol{A}$ to any state $\psi_{n\ell m} = Y_{\ell m}(\theta, \varphi) f_{n\ell}(r)/r$, without the restriction $m = \ell$ of Grosse [13] and Granados [14]. This procedure allows us to obtain the radial ladder operators for $f_{n\ell}(r)$, acting on ℓ, and therefore it shows that they are restrained in $\boldsymbol{A}$. We arrive at this conclusion by imposing our results to be consistent with those derived from the Wigner–Eckart theorem. We stress the fact that no spin operators are needed in our formalism. Moreover, we also note that such deduction does not use either the Dirac radial decomposition or the particular state $\psi_{n\ell\ell}$ of the system.

In section 3, by means of a simple argument, we show how the ladder operators obtained in section 2 must coincide with those of the IH factorization method, to be consistent with the radial Schrödinger equation. Also in this section, we show that the form of the known superpotentials of the problem is connected with one of the necessary conditions, in order to apply the IH factorization method. In section 4 we find the constraints for the quantum numbers n and ℓ so that the corresponding ladder operators are well defined. Finally, in section 5, we give some concluding remarks.

2. The operators $A_\pm$, A_0 and ladder operators for $f_{n\ell}(r)$, acting on ℓ

It is well known [1] that the Hermitian LRLV

$$\boldsymbol{A} = \frac{1}{\sqrt{2\mu E}} \left\{ \frac{1}{2}(\boldsymbol{L} \times \boldsymbol{p} - \boldsymbol{p} \times \boldsymbol{L}) + \mu e^2 \frac{\boldsymbol{r}}{r} \right\} \tag{1}$$

represents the constant of motion associated with the dynamical symmetry of the non-relativistic quantum problem with potential $-e/r$. Here $\boldsymbol{p}$ ($\boldsymbol{L}$) is the linear (angular) momentum operator for a spinless particle of mass μ. Equation (1) is equivalent to

$$A_i = \frac{1}{\sqrt{2\mu E}} \left\{ -x_i \boldsymbol{p}^2 + p_i \boldsymbol{r} \cdot \boldsymbol{p} + \mu e^2 \frac{x_i}{r} \right\} \qquad i = 1, 2, 3. \tag{2}$$

This implies that

$$A_\pm = \frac{1}{\sqrt{2\mu E}} \left\{ (x \pm \mathrm{i}y) \left(-\boldsymbol{p}^2 + \frac{\mu e^2}{r} \right) + (p_x \pm \mathrm{i}p_y) \boldsymbol{r} \cdot \boldsymbol{p} \right\} \tag{3}$$

and

$$A_0 \equiv A_z = \frac{1}{\sqrt{2\mu E}} \left\{ z \left(-\boldsymbol{p}^2 + \frac{\mu e^2}{r} \right) + p_z \boldsymbol{r} \cdot \boldsymbol{p} \right\}. \tag{4}$$

We want to apply the spherical components $A_\pm$ and A_0 to any element of the standard basis, $\psi_{n\ell m}$, of the wavefunction space of a spinless particle $\xi_{\boldsymbol{r}}$. Any central-potential wavefunction is necessarily [9] of the form

$$\psi_{n\ell m} = Y_{\ell m}(\theta, \phi) R_{n\ell}(r). \tag{5}$$

This state is an eigenfunction of the constants of motion H, L^2 and L_z. $Y_{\ell m}(\theta,\phi) = \frac{1}{\sqrt{2\pi}}P_{\ell m}(\theta)\mathrm{e}^{\mathrm{i}m\phi}$ is a spherical harmonic, and $R_{n\ell}(r)$ satisfies the radial Schrödinger equation

$$\left[\frac{\hbar^2}{\mu}H_\ell\right] rR_{n\ell}(r) = E_{n\ell} rR_{n\ell}(r) \tag{6}$$

where

$$2H_\ell \equiv -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\ell(\ell+1)}{r^2} - \frac{2}{a_0 r} \equiv -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V(r,\ell). \tag{7}$$

$V(r,\ell)$ denotes the effective potential of the problem, and $a_0 \equiv \hbar^2/\mu e^2$.

By expressing the operators $(x \pm \mathrm{i}y)$, $\boldsymbol{p}^2$ and $\boldsymbol{r}\cdot\boldsymbol{p}$ in spherical coordinates, we find

$$(x \pm \mathrm{i}y) = r\sin\theta\mathrm{e}^{\pm\mathrm{i}\phi} \tag{8}$$

$$\boldsymbol{p}^2 = -\hbar^2\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \hbar^2\frac{\ell(\ell+1)}{r^2} \tag{9}$$

$$\boldsymbol{r}\cdot\boldsymbol{p} = -\mathrm{i}\hbar r\frac{\partial}{\partial r}. \tag{10}$$

On the other hand, we have that

$$p_x \pm \mathrm{i}p_y = -\mathrm{i}\hbar\left(\frac{\partial}{\partial x} \pm \mathrm{i}\frac{\partial}{\partial y}\right). \tag{11}$$

By substituting equations (8)–(11) into (3), we obtain

$$A_\pm = \frac{1}{\sqrt{2\mu E}}\left\{\sin\theta\mathrm{e}^{\pm\mathrm{i}\phi}\left[\hbar^2\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\hbar^2\ell(\ell+1)}{r} + \mu e^2\right] - \hbar^2\left(\frac{\partial}{\partial x} \pm \mathrm{i}\frac{\partial}{\partial y}\right) r\frac{\partial}{\partial r}\right\}. \tag{12}$$

Substituting equations (9) and (10) into (4) leads us to

$$A_0 = \frac{1}{\sqrt{2\mu E}}\left\{\cos\theta\left[\hbar^2\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\hbar^2\ell(\ell+1)}{r} + \mu e^2\right] - \hbar^2\left(\frac{\partial}{\partial z}\right) r\frac{\partial}{\partial r}\right\}. \tag{13}$$

We define $f_{n\ell}(r)$, $g^-_{\ell+1}$ and g^+_ℓ as

$$f_{n\ell}(r) \equiv rR_{n\ell}(r) \tag{14}$$

$$g^-_{\ell+1} \equiv -\frac{\mathrm{d}}{\mathrm{d}r} + u(r,\ell+1) \tag{15}$$

and

$$g^+_\ell \equiv \frac{\mathrm{d}}{\mathrm{d}r} + u(r,\ell) \tag{16}$$

with

$$u(r,\ell+1) \equiv \frac{\ell+1}{r} - \frac{1}{(\ell+1)a_0}. \tag{17}$$

Equation (14) defines what we have called the reduced radial wavefunction.

By applying the operators $A_\pm$ and A_0 to $\psi_{n\ell m}$, and using recursion relations for spherical harmonics [15], we obtain

$$A_\pm\psi_{n\ell m} = \frac{\hbar^2}{\sqrt{2\mu E}}\left\{(\ell+1)\alpha^\pm Y_{\ell+1,m\pm1}\left(-\frac{1}{r}g^-_{\ell+1}\right)f_{n\ell}(r) + \ell\beta^\pm Y_{\ell-1,m\pm1}\left(-\frac{1}{r}g^+_\ell\right)f_{n\ell}(r)\right\} \tag{18}$$

and

$$A_0\psi_{n\ell m} = -\frac{\hbar^2}{\sqrt{2\mu E}}\left\{(\ell+1)\gamma Y_{\ell+1,m}\left(-\frac{1}{r}g^-_{\ell+1}\right)f_{n\ell}(r) + \ell\epsilon Y_{\ell-1,m}\left(\frac{1}{r}g^+_{\ell}\right)f_{n\ell}(r)\right\} \qquad (19)$$

where

$$\alpha^{\pm} = \pm\sqrt{\frac{(\ell\pm m+1)(\ell\pm m+2)}{(2\ell+1)(2\ell+3)}} \qquad \beta^{\pm} = \mp\sqrt{\frac{(\ell\mp m)(\ell\mp m-1)}{(2\ell+1)(2\ell-1)}}$$

$$\gamma = \sqrt{\frac{(\ell+m+1)(\ell-m+1)}{(2\ell+3)(2\ell+1)}} \qquad \text{and} \qquad \epsilon = \sqrt{\frac{(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)}}.$$

From equations (18) and (19) and the following argument we can recognize the operators $g^-_{\ell+1}$ and g^+_ℓ as ladder operators for $f_{n\ell}(r)$, acting on ℓ.

It is a general result [17] that if G is an element of the symmetry group of any problem, then

$$GH = HG.$$

By using this fact and applying the element G to the stationary Schrödinger equation, we obtain

$$H(G\psi_{n\alpha}) = E_n(G\psi_{n\alpha}).$$

Thus, $G\psi_{n\alpha}$ is a solution of the Shrödinger equation corresponding to the eigenvalue E_n. Therefore, this function can be expanded in terms of the eigenfunctions $\psi_{n\alpha}$ as follows:

$$G\psi_{n\alpha} = \sum_\beta \psi_{n\beta}A_{\beta\alpha}. \qquad (20)$$

We note that equations (18) and (19)—as well as (23) and (24)—are particular cases of this relation, being combinations of eigenstates of $H_{\ell+1}$ and $H_{\ell-1}$, pertaining to the subspaces $\xi(\ell+1, m)$ and $\xi(\ell-1, m)$, respectively. In fact $A_\pm\psi_{n\ell m}$ and $A_0\psi_{n\ell m}$ are eigenfunctions of H_ℓ, which implies that $E_{n\ell} = E_{n\ell+1} = E_{n\ell-1}$. This shows the connection between the dynamical symmetry of the problem and its accidental degeneracy. Then the spherical components of $\boldsymbol{A}$ connect all the states with a given energy. Hence the radial operators $g^-_{\ell+1}$ and g^+_ℓ in equations (18) and (19) do not change the index n of the function $f_{n\ell}(r)$.

Notice that in equation (18) ((19)) the spherical harmonics $Y_{\ell+1,m\pm1}$ ($Y_{\ell+1,m}$) and $Y_{\ell-1,m\pm1}$ ($Y_{\ell-1,m}$) are contained in the first and second terms of $A_\pm\psi_{n\ell m}$ ($A_0\psi_{n\ell m}$) respectively. From the first term of equation (18) or (19), together with (20), we immediately conclude that the operator $g^-_{\ell+1}$ acting over $f_{n\ell}(r)$ gives as result only a change in the subindex ℓ, from ℓ to $\ell+1$. This is because the second subindex in $f_{n\ell}(r)$ must coincide with the first one of the spherical harmonic $Y_{\ell+1,m\pm1}$ or $Y_{\ell+1,m}$. Formally this means that

$$g^-_{\ell+1}f_{n\ell}(r)\,\alpha\, f_{n\ell+1}(r). \qquad (21)$$

Similarly from equation (20) and the second term of equation (18) or (19) it follows that

$$g^+_\ell f_{n\ell}(r)\,\alpha\, f_{n\ell-1}(r). \qquad (22)$$

The relationships (21) and (22) are confirmed by comparing (18) and (19) with the well known expressions [11]

$$A_\pm\psi_{n\ell m} = \mp\alpha(\pm\alpha^\pm)\psi_{n,\ell+1,m\pm1} \pm \beta(\mp\beta^\pm)\psi_{n,\ell-1,m\pm1} \qquad (23)$$

and

$$A_0\psi_{n\ell m} = \alpha\gamma\;\psi_{n,\ell+1,m} + \beta\epsilon\psi_{n,\ell-1,m} \qquad (24)$$

which follow from the application of the Wigner–Eckart theorem, with

$$\alpha = \sqrt{(n+\ell+1)(n-\ell-1)} \qquad \text{and} \qquad \beta = \sqrt{(n+\ell)(n-\ell)}.$$

We note from equation (23) (equation (24)) that $A_\pm\psi_{n\ell m}$ ($A_0\psi_{n\ell m}$) is a combination of $\psi_{n\ell+1m\pm1}$ and $\psi_{n\ell-1m\pm1}$ ($\psi_{n\ell+1m}$ and $\psi_{n\ell-1m}$), which is not true if we only consider the radial part of the states.

We can say about the recurrence relations (21) and (22) that:

(a) These relations cannot be a consequence of the Wigner–Eckart theorem, but of the explicit application of $A_\pm$ and A_0 to any element $\psi_{n\ell m}$. We also note that the condition $m = \ell$ that was imposed in [13] is superfluous.

(b) The first (second) of them transforms the radial part of any state in the subspace $\xi(\ell, m)$ of H_ℓ into the radial part of the corresponding state in the subspace $\xi(\ell+1, m)$ ($\xi(\ell-1, m)$) of $H_{\ell+1}$ ($H_{\ell-1}$). This suggests the origin of the supersymmetric approach to the radial problem. However, according to equations (15) and (16), the operators g_0^- and g_0^+ are not defined. Because of this if $\ell = -1$ we have that $g^-_{(-1)+1} f_{n(-1)} \equiv g_0^- f_{n(-1)}$ cannot be f_{n0} in spite of equation (21). Analogously, if $\ell = 0$ then $g_0^+ f_{n0}$ cannot be equal to $f_{n(-1)}$ in spite of equation (22). This implies that there is no ladder operator that transforms $f_{n0}(r)$ into any state with $\ell < 0$ (if it existed). Conversely, there is no ladder operator that transforms a possible state with $\ell < 0$ into to the state $f_{n0}(r)$. Therefore the structure of the operators in equations (15) and (16) enables us to conclude that

$$\ell_{\min} = 0 \tag{25}$$

which is a standard result.

Incidentally, we note that equations (18) and (19) are still valid when $\ell = 0$ because their term that contains g_0^+ is proportional to ℓ.

(c) We can combine them to obtain the relations

$$g^+_{\ell+1} g^-_{\ell+1} f_{n\ell}(r) \,\alpha\, f_{n\ell}(r) \qquad g^-_\ell g^+_\ell f_{n\ell}(r) \,\alpha\, f_{n\ell}(r)$$

or equivalently

$$(g^+_{\ell+1} g^-_{\ell+1} - k_1) f_{n\ell}(r) = 0 \tag{26}$$

$$(g^-_\ell g^+_\ell - k_2) f_{n\ell}(r) = 0 \tag{27}$$

where k_1 and k_2 are constants that we will determine in the following section. The operators in equations (26) and (27) are related to the radial Hamiltonian of the problem, H_ℓ, by the equalities

$$g^+_{\ell+1} g^-_{\ell+1} = 2(H_\ell + \eta_\ell) \tag{28}$$

$$g^-_\ell g^+_\ell = 2(H_\ell + \eta_{\ell-1}) \tag{29}$$

with

$$2\eta_\ell = \left(\frac{1}{(\ell+1)a_0}\right)^2. \tag{30}$$

As a final comment, we note that according to equations (18) and (19) the operators $A_\pm$ and A_0 can be written as

$$\begin{pmatrix} A_\pm \\ A_0 \end{pmatrix} = \frac{\hbar^2}{\sqrt{2\mu E}} \frac{1}{\sqrt{2\ell+1}} \left[\frac{\ell+1}{\sqrt{2\ell+3}} \begin{pmatrix} \mathfrak{A}^\dagger_\pm \\ -\mathfrak{A}^\dagger_0 \end{pmatrix} \left(-\frac{1}{r} g^-_{\ell+1}\right) + \frac{\ell}{\sqrt{2\ell-1}} \begin{pmatrix} \mathfrak{A}_\mp \\ \mathfrak{A}_0 \end{pmatrix} \left(-\frac{1}{r} g^+_\ell\right) \right]$$

where the operators $\mathfrak{A}^\dagger_\pm$, $\mathfrak{A}_\pm$ and $\mathfrak{A}_0$ defined in [16] are related to the operators of the $o_{3,2}$ algebra associated with spherical harmonics.

3. Relation between g operators, the IH ladder operators and SUSY charges

Starting from the fact that $f_{n\ell}(r)$ satisfies the equation (6)

$$(2H_\ell - \lambda) f_{n\ell}(r) = 0 \tag{31}$$

with $\lambda = 2\mu E/\hbar^2$ (for $E < 0$).

Next, the substitution of equations (28) and (29) into (26) and (27) leads us to

$$[2(H_\ell + \eta_\ell) - k_1] f_{n\ell}(r) = 0 \tag{32}$$

$$[2(H_\ell + \eta_{\ell-1}) - k_2] f_{n\ell}(r) = 0. \tag{33}$$

Since equations (31)–(33) must be equivalent, we find that

$$k_1 = \lambda + 2\eta_\ell \tag{34}$$

$$k_2 = \lambda + 2\eta_{\ell-1}. \tag{35}$$

Therefore equations (32) and (33) take the form

$$g^+_{\ell+1} g^-_{\ell+1}\ f_{n\ell}(r) = [\lambda + 2\eta_\ell]\ f_{n\ell}(r) \tag{36}$$

$$g^-_\ell\, g^+_\ell f_{n\ell}(r) = [\lambda + 2\eta_{\ell-1}]\ f_{n\ell}(r) \tag{37}$$

which are correct if relations (21) and (22) satisfy the equalities

$$g^-_{\ell+1} f_{n\ell}(r) = \sqrt{\lambda + 2\eta_\ell}\, f_{n\ell+1}(r) \tag{38}$$

$$g^+_\ell f_{n\ell}(r) = \sqrt{\lambda + 2\eta_{\ell-1}}\, f_{n\ell-1}(r). \tag{39}$$

The operators defined in equations (15) and (16) that satisfy (38) and (39) are identical to those obtained by means of the IH-factorization method. In fact, they reproduce the results of the class I type F IH factorization for the Kepler–Coulomb (KC) problem [3]. In the previous section we showed that these operators can be deduced by using an independent and more fundamental procedure than that used by other authors [7, 10, 11].

On the other hand [18], a Hamiltonian of the type

$$H = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V(r) \tag{40}$$

would admit only one of the forms

$$H^{(0)} = Q^+Q^- \qquad \text{or} \qquad H^{(1)} = Q^-Q^+ \tag{41}$$

where

$$Q^\pm = \frac{1}{2}\left(\mp\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\mathrm{d}\chi}{\mathrm{d}r}\right) \tag{42}$$

and $\chi = \chi(r)$, only if $F = \mathrm{d}\chi/\mathrm{d}r \equiv \chi'$ satisfies the Riccati equation

$$V(r) = \begin{cases} F^2 - F' & \text{if} \quad H \text{ admits the form } H^{(0)} \\ F^2 + F' & \text{if} \quad H \text{ admits the form } H^{(1)}. \end{cases} \tag{43}$$

$H^{(0)}$ and $H^{(1)}$ are the two isospectral Hamiltonians, except in the ground state. They can be viewed as the bosonic and the fermionic components of the matrix SUSY Hamiltonian

$$\mathcal{H} = \begin{pmatrix} H^{(1)} & 0 \\ 0 & H^{(0)} \end{pmatrix} = \tfrac{1}{2}[(p_r^2 + F^2)I + F'\sigma_3]$$

in which I is the unit matrix and σ_3 is the Pauli spin matrix [4]. The quantity $F(r)$ is generally referred to as the *superpotential* in SUSYQM literature [5].

Since the energy of the hydrogen atom ground state is not zero, its Hamiltonian H_ℓ does not admit any of the forms $H^{(0)}$ or $H^{(1)}$ [19]. However, we observe that the equations

$$F^2 - F' = V(r,\ell) + 2\eta_\ell \equiv 2V^{(0)} \tag{44}$$

$$F^2 + F' = V(r,\ell+1) + 2\eta_\ell \equiv 2V^{(1)} \tag{45}$$

are satisfied simultaneously by

$$F \equiv -u(r,\ell+1) = \frac{1}{(\ell+1)a_0} - \frac{\ell+1}{r}. \tag{46}$$

In fact, the superpotential F satisfies simultaneously equations (44) and (45).

Because η_ℓ is not a function of r and it appears in $V^{(0)}$ and $V^{(1)}$ with the same sign, the nonlinear differential equations (44) and (45) imply the relation

$$u(r,\ell+1) = \frac{1}{2}\frac{V'(r,\ell) + V'(r,\ell+1)}{V(r,\ell) - V(r,\ell+1)} \tag{47}$$

where $V' = \mathrm{d}V/\mathrm{d}r$.

We recognize equation (47) as one of the necessary conditions for the IH factorization [3] of the Schrödinger equation, (31). Such factorization is confirmed by the existence of equations (38) and (39), whose effect in equations (28) and (29) is to eliminate all reference to η_ℓ, allowing us to recover the radial Schrödinger equation of the hydrogen atom.

As we have already said in section 2, equations (28) and (29) naturally lead to the well known pair of isospectral Hamiltonians of the problem [20]. In fact, from equation (28)

$$(H_\ell + \eta_\ell) = \left(\frac{1}{\sqrt{2}}g^+_{\ell+1}\right)\left(\frac{1}{\sqrt{2}}g^-_{\ell+1}\right) \equiv Q^+_\ell Q^-_\ell. \tag{48}$$

From equations (7) and (44) it follows that

$$Q^+_\ell Q^-_\ell = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V^{(0)} \equiv H^{(0)} \tag{49}$$

which is a Hamiltonian of the form given by equation (40). Analogously, by making the change $\ell \to \ell + 1$ in equation (29) and using (7) and (45), the associated Hamiltonian is

$$H^{(1)} \equiv Q^-_\ell Q^+_\ell = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V^{(1)}. \tag{50}$$

From equations (49) and (50) we can see that the effective potentials for two consecutive values of ℓ determine the supersymmetric partner potentials $V^{(0)}$ and $V^{(1)}$, from which the hydrogen atom spectrum can be obtained. For this purpose the restriction $n \geqslant \ell + 1$ is imposed *a priori* by other authors [20, 21]. We will see in the next section how this restriction can be deduced with the help of the ladder operators for $f_{n\ell}(r)$ acting on n.

The basic idea to obtain such a spectrum is to use the fact that the application of $H^{(0)}$ to the state with maximum admissible value of ℓ, consistent with n, satisfies that

$$H^{(0)} f_{n\,\ell_{\max}} \equiv Q^+_\ell (Q^-_\ell f_{n\,\ell_{\max}}) = 0. \tag{51}$$

This relation implies the existence of a state of the hydrogen atom spectrum that serves as an eigenstate with a vanishing eigenvalue for the Hamiltonian $H^{(0)}$. The remaining part of the KC spectrum is obtained by means of the successive application of $Q^+_{\ell-1}$ [20].

4. Restrictions to the quantum numbers n and ℓ

Equation (31) can be rewritten as

$$\left\{-r^2\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\left[-\lambda r^2-\frac{2}{a_0(-\lambda)^{1/2}}(-\lambda)^{1/2}r+\ell(\ell+1)\right]\right\}f_{n\ell}(r)=0 \tag{52}$$

that suggests the form

$$(O_n+\ell(\ell+1))f_{n\ell}(r)=0 \tag{53}$$

where O_n is defined as

$$O_n=-r^2\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\left[\frac{\sigma^2r^2}{4}-n\sigma r\right]=-r^2\frac{\mathrm{d}^2}{\mathrm{d}r^2}+U(r,n) \tag{54}$$

with

$$n\equiv a_0^{-1}(-\lambda)^{-1/2}=\frac{2\mu e^2}{\sigma\hbar^2}$$

and

$$\sigma^2\equiv-\frac{8\mu E}{\hbar^2}=-4\lambda \qquad (E<0).$$

We denote the ladder operators for the reduced radial wavefunction $f_{n\ell}(r)$, acting on n, by φ_n^+ and φ_n^- such that

$$\varphi_n^+f_{n\ell}(r)\;\alpha\;f_{n+1\ell}(r) \tag{55}$$

$$\varphi_n^-f_{n\ell}(r)\;\alpha\;f_{n-1\ell}(r). \tag{56}$$

These operators have been derived following a method more general than the IH-factorization method [22], and are given by

$$\varphi_n^+=\frac{r}{na_0}-n-r\frac{\mathrm{d}}{\mathrm{d}r} \tag{57}$$

$$\varphi_n^-=\frac{r}{na_0}-n+r\frac{\mathrm{d}}{\mathrm{d}r}. \tag{58}$$

The operators $\varphi_n^\pm$ are formally defined by

$$n\neq 0 \tag{59}$$

which is consistent with the non-existence of the state $f_{0\ell}(r)$ in the hydrogen atom spectrum. Furthermore, from equations (15), (17) and (51) the state $f_{n\,\ell_{\max}}(r)$ satisfies

$$g^-_{\ell_{\max}+1}f_{n\,\ell_{\max}}=\left(-\frac{1}{(\ell_{\max}+1)a_0}+\frac{\ell_{\max}+1}{r}-\frac{\mathrm{d}}{\mathrm{d}r}\right)f_{n\,\ell_{\max}}(r)=0 \tag{60}$$

which, according to equation (58), is formally equivalent to the expression

$$-\frac{1}{r}(\varphi^-_{\ell_{\max}+1})f_{n\,\ell_{\max}}(r)=0. \tag{61}$$

Evidently equation (61) is a particular case of (56). From this we conclude that the subindex of $\varphi^-_{\ell_{\max}+1}$ must be equal to the first subindex of $f_{n\,\ell_{\max}}$. This means that $n=\ell_{\max}+1$, or

$$\ell_{\max}=n-1. \tag{62}$$

Equations (25), (59) and (62) lead us to

$$0\leqslant\ell\leqslant n-1 \qquad \text{and} \qquad n>0$$

which, as we saw above in this section, can be obtained by imposing on both $f_{n\ell}(r)$ radial ladder operators (those that act on ℓ and those that act on n) the condition to be well defined, along with the existence of the so-called ground state $f_{n\,\ell_{\max}}$ [20] that satisfies equations (60) and (61).

5. Conclusion

We have found for the KC problem that there exists a straightforward connection between the constants of motion associated with the dynamical symmetry, the radial factorization of the problem and its supersymmetric treatment. This explicit connection has been directly found when the spherical components of the LRLV act on any arbitrary element in the standard basis of the state space ξ_r. Therefore, our procedure turns out to be more fundamental than those used in other papers, since it allows us to reproduce independently the results of the IH-factorization method, as well as those of the supersymmetric approach that admits the radial part of the problem.

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